SOLUTION EXERCISE SHEET 19

Exercise 1. Recall that if f is a meromorphic function with a pole at z_0 of order k, then near that pole it has a Laurent series expansion given by

$$f(z) = \sum_{j=-k}^{\infty} a_j (z - z_0)^k$$

and the residue is given by a_{-1}. Moreover, we can recast f as

$$f(z) = (z - z_0)^{-k} \sum_{j=-k}^{\infty} a_j (z - z_0)^{k+j}$$

Finally, since the series $\sum_{j=-k}^{\infty} a_j (z - z_0)^{k+j}$ converges locally uniformly, we can exchange differentiation with summation to compute that

$$\frac{d^k}{dz^k}((z - z_0)^k f(z)) = (k-1)! a_{-1} + \sum_{j=1}^{\infty} b_j (z - z_0)^j$$

for some computable coefficients b_j and the desired formula follows.

Exercise 2. Assume for a contradiction that the limit exists and is finite. We set $g(x) = \frac{f(x)}{x}$ and employ the change of variables $x = e^y$. Then the convergence of the limit

$$\lim_{M \to \infty} \int_1^M \frac{f(x) - x}{x^2} dx \tag{1}$$

is equivalent to the convergence of

$$\lim_{M \to \infty} \int_0^{\log(M)} g(e^y) - 1 dy. \tag{2}$$

Note that for (2) to exist and be finite, we must necessarily have that for every $a > 0$ fixed and every $\varepsilon > 0$ there exists a r_0 such that for all $r \geq r_0$ we have that

$$\left| \int_{r-a}^{r} g(e^y) - 1 dy \right| < \varepsilon. \tag{3}$$

Now, since $\liminf_{x \to \infty} g(x) < 1$ there exist a $\delta > 0$ and a subsequence $\{x_n\}_{n=0}^{\infty}$ such that $g(x_n) < 1 - \delta$. for all $n \in \mathbb{N}$. We now set $c = \frac{\delta}{2(1-\delta)}$ and consider the intervals

$$I_n := [\frac{x_n}{1+c}, x_n].$$

Then, since f is monotonically increasing the inequality

$$g(x) \leq \frac{f(x_n)}{x} \leq \frac{x_n g(x_n)}{x} \leq (1+c) g(x_n) \leq \left(1 + \frac{\delta}{2(1-\delta)}\right)(1 - \delta) = 1 - \frac{\delta}{2}$$

holds for all $x \in I_n$. Consequently, we obtain that

$$\int_{\frac{x_n}{1+c}}^{x_n} \frac{g(x)-1}{x} dx = \int_{\log(x_n)-\log(1+c)}^{\log(x_n)} g(e^y) - 1 dy \leq - \int_{\log(x_n)-\log(1+c)}^{\log(x_n)} \frac{\delta}{2} dy$$
$$= -\frac{\delta}{2} \log(1+c).$$

This visibly contradicts (3) which concludes the proof.

Exercise 3. Recall the geometric series formula

$$\frac{1}{1-p^{-s}} = \sum_{j=0}^{\infty} p^{-js}.$$

Further, for notational convenience, we let $D(p_*)$ be the set of all natural numbers such that all prime divisors are smaller than p_* and, for any p_* fixed, we let $p_1, \dots p_k$ be all prime numbers that are smaller than p_*. Then, given that the product of absolutely convergent series is again absolutely convergent, we can rearrange to infer

$$\prod_{1<p<p_*} \frac{1}{1-p^{-s}} = \prod_{1<p<p_*} \sum_{j=0}^{\infty} p^{-js} = \sum_{j_1,\dots j_k=0}^{\infty} p^{-j_1 s} \cdot \dots \cdot p^{-j_k s} = \sum_{n \in D(p_*)} \frac{1}{n^s}$$

Exercise 4. We rewrite

$$\frac{\log p}{p^s(p^s-1)} = \frac{\log p}{p^{2s}(1-p^{-s})}$$

Therefore, since $|1-p^{-s}| \geq 1 - \frac{1}{\sqrt{2}} =: c$ we see that

$$\left| \frac{\log p}{p^s(p^s-1)} \right| \leq c^{-1} \frac{\log p}{p^{2\operatorname{Re}(s)}}$$

Thus, since the series

$$c^{-1} \sum_{n=1}^{\infty} \frac{\log n}{n^{2s}}$$

converges normally for $\operatorname{Re} s > \frac{1}{2}$ and dominates

$$\sum_{p\, prime} \frac{\log p}{p^s(p^s-1)}$$

the normal convergence and with it the holomorphicity, follows. Lastly, we want to show that we cannot extend

$$F(z) := \sum_{p\, prime} \frac{\log p}{p^z(p^z-1))}$$

holomorphically beyond the threshold $\operatorname{Re} z = \frac{1}{2}$. To that end, we assume for a contradiction that F extends holomorphically to the set $H_{\frac{1}{2}-\delta} := \{z \in \mathbb{C} : \operatorname{Re} z > \frac{1}{2} - \delta\}$ for some $\delta > 0$. This would in particular imply that F is bounded on some small ball around $z = \frac{1}{2}$. To see that this is not possible, we note that for $x \in \mathbb{R}$ with

$\operatorname{Re} x > \frac{1}{2}$ we have

$$\frac{\log p}{p^x(p^x-1)} \geq \frac{1}{1-\sqrt{2}} p^{-2x}.$$

So, we need to show that

$$P(s) := \sum_{p\, prime} p^{-s}$$

does not remain bounded in any neighborhood of $s = 1$. From Exercise 3 and the Taylor expansion of the function log(1+x) we infer

$$\begin{aligned}
\log\left(\sum_{n\in D(p_*)} \frac{1}{n^s}\right) &= \log\left(\prod_{1\leq p<p_*} \frac{1}{1-p^{-s}}\right) = -\sum_{1\leq p<p_*} \log\left(1-p^{-s}\right) \\
&= \sum_{1\leq p<p_*} \sum_{j=1}^{\infty} \frac{1}{jp^{js}} \\
&= \sum_{1\leq p<p_*} \frac{1}{p^s} + \sum_{1\leq p<p_*} \sum_{j=2}^{\infty} \frac{1}{jp^{js}}.
\end{aligned}$$

Next, we compute that

$$\begin{aligned}
\sum_{1\leq p<p_*} \sum_{j=2}^{\infty} \frac{1}{jp^{js}} &\leq \frac{1}{2} \sum_{1\leq p<p_*} \sum_{j=2}^{\infty} \frac{1}{p^{js}} = \frac{1}{2} \sum_{1\leq p<p_*} p^{-2s} \sum_{j=0}^{\infty} \frac{1}{p^{js}} \\
&= \frac{1}{2} \sum_{1\leq p<p_*} p^{-2s} \frac{1}{1-p^{-s}} \leq \frac{1}{2} \sum_{1\leq p<p_*} p^{-2} \frac{1}{1-p^{-1}} \\
&\leq \sum_{1\leq p<p_*} p^{-2} \\
&\leq C
\end{aligned}$$

where C can be chosen independently of p_* and $s \in [1, \infty)$. Therefore, we see that we have the inequality

$$\log\left(\sum_{n\in D(p_*)} \frac{1}{n^s}\right) = \sum_{1\leq p<p_*} \frac{1}{p^s} + f(s, p_*)$$

for some f that stays uniformly bounded on $[1, \infty) \times \mathbb{N}$. Finally, from the divergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

it follows that we can make the left hand side arbitrarily large by letting $s \to 1$ and $p_* \to \infty$.